A Process Oriented Definition of Number

David Rolfe

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Abstract

In this paper Russell's definition of number is criticized. Russell's assertion that a number is a particular kind of set implies that number has the properties of a set. It is argued that this would imply that a number contains elements and that this does not conform to our intuitive notion of number. An alternative definition is presented in which number is not seen as an object, but rather as a process and is related to the act of counting and is tightly bound up with the idea of time. Working from the idea that the description of a thing is not the thing itself, it is argued that a function should not be seen as a subset of the Cartesian product of two sets but can be described in this way. Number is then defined as a particular type of bijective function rather than a set. Definitions of equality and addition are developed. In defining addition an interesting error in Russell's definition of addition is corrected

The idea of number has grown and changed over the millennia. The ancient Greeks discovered that their collection of rational numbers was incomplete in the sense that it did not contain the answers to questions that naturally derive from them. So the square root of two is not rational but is part of what we now call the algebraic numbers. In the nineteenth century this collection was found to be similarly incomplete and the real numbers were developed. Finally the complex numbers were developed and found to be algebraically complete. By this time the notion of number had traveled far from its origins in simple counting, but all subsequent ideas about number are to some extent derivative of the counting numbers and so efforts were made to develop an entirely formal description of them.

Peano developed a set of axioms that were initially thought to describe the necessary and sufficient conditions for number. The axioms are as follows.[1]

1. 0 is a number
2. The successor of any number is a number
3. No two numbers have the same successor
4. 0 is not the successor of any number
5. Any property that belongs to x_0 and to x_{n+1} providing it belongs to x_n belongs to all x.

Unfortunately these axioms were shown to be necessary but not sufficient and many objects that conformed to the axioms were clearly not numbers.

Russell gives a good example of this:

Let 0 be the number 1 and successor mean half. This generates the set of "numbers" $\left(1, \frac{1}{2}, \frac{1}{4}, \ldots\right)$. This is clearly not what we want.

There are two ways to resolve this problem. One would be to discover another axiom would make them both necessary and sufficient. The other would be to unambiguously characterize one set of objects that satisfies the axioms. This is the way Russell chooses to attack the problem. His attempt is described and criticized and an alternative is proposed.

Bertrand Russell uses the ideas of a set of sets and one to one correspondence or a bijective function to define number. It is very easy to

[1] Bertrand Rusell, Introduction to Mathematical Philosophy (Digireads.com Publishing) , 10

fall into circularity when following Russell's program. One must avoid speaking of sets with the same number of elements when trying to define number! Russell is careful to do this and is successful. He asks us to consider sets, between which there is a bijection. This means that for every element in the first set there is an associated element in the second set and no two distinct elements of the first set are associated with the same element in the second set. Furthermore each element of the second set has an element of the first set with which it is associated. Two sets, for which such an association holds, are called similar.

One could then consider a collection of similar sets, that is, any two sets in the collection would be similar to each other. [2] A given number, such as five, would then be equivalent to a collection of similar sets and number in general would be defined to be the set of all such sets or collections.

When this definition is presented one might feel a certain uneasiness. The precise source of this uneasiness will be discussed shortly. But even Russell acknowledges a certain dissatisfaction. In his Introduction to Mathematical Philosophy he states:[3]

> *It [the class of couples] is indubitable and not difficult to define. Whereas the number 2 in any other sense, is a metaphysical entity about which we can never feel sure that it exists or that we have tracked it down. It is therefore more prudent to content ourselves with the class of couples, which we are sure of, than to hunt for a problematical number 2 which must always remain elusive.*

This is a curious and important admission by Russell. He admits the definition he proposes is problematic, but then goes on to say it is the best anyone can do. It was certainly the best Russell could do at the time, but I

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[2] This similarity relation is, in fact, an equivalence relation and so for any collection of similar sets the following holds. It S is the similar relation then

1. aSa for all a in the collection
2. aSb implies bSa
3. If aSb and bSc then aSc

[3] Russell, 15

would not give up entirely on the project of improving the definition as Russell does. Nor in his attempt to banish metaphysics from the discussion, does Russell admit that, in fact, a set is most emphatically a metaphysical object. He is trying to avoid the ontological questions that concern abstract ideas, but is nonetheless caught up in it, even if he declines to discuss it.

What is the source of the unease to which Russell refers? I think is has to do with the qualities or properties of a thing. If two things are the same sort of thing, then they should have the same properties. So if two animals are house cats, we can be confident that the share certain properties. They are both mammals, are of a certain size, have retractable claws, and so forth. To say that a number is a certain kind of set would be to imply that the number and the set share certain properties. What is a property that all sets have? They contain elements. [4] In what sense could it be said that a given number contains elements? This is rather like asking what color is a C major chord. Numbers are not collections. To follow Russell's program and avoid all that metaphysical vagueness, one must be prepared to accept a definition of number that does not seem to conform to what we would normally think a number is.

In the following discussion we will not go far from Russell's ideas, but will not equate a number with a set. When considering what a number might be we are naturally lead to considering what we do when we count. We were given, in grade school, a very handy set. It is called the counting numbers, and when we count objects, "1,2,3, …" we are creating a correspondence between the set of objects and the counting numbers. So we say there are 5 objects when we get up to 5 and run out of objects. We need to find a way to capture this idea with out actually referring to numbers. Otherwise we fall into the circularity discussed above.

In addition the intuitive notion of number that we have is already really quite abstract. We do not insist that the number that measures geese is different from that that measures fish. It is property that all collections

[4] Except the null set of course

share. As we develop a formal definition of number we will see that this level of abstraction becomes quite explicit.

For any give set, there exists a bijective function that maps the elements of the set onto itself. In particular there is the identity function that maps each element of a set onto itself. [5] If one thinks of this function as operating on the elements of the set one at a time, the function can be seen as a kind of counting and this bijection can be thought of as the number of elements in the set. In this case the full generality of number is not yet developed as each set had its own identity function and so each type of collection has its own number. This conforms with the primitive notion of number in which different kinds of collections get there own descriptor.

Before we continue it is important to address the idea that a function can be viewed as a set. A function is sometimes defined as a subset of the cross product of two sets or a set with itself. If this view is accepted then equating the idea of a number with a particular type of function leads us back to saying that a number is a set. But the reasoning that argues against numbers being sets works as well for functions. It is true that a set of ordered pairs could be used to define the action of a function, but that does not mean that it is identical to the function. The description of a thing is not identical to the thing.

So if (a,b) is in some set, F, we could take this to mean that a is mapped into b under some function. But it is the function that does the mapping. In this view functions are active processes that act in time to move between elements of sets or to associate one element of a set with another. Sets are objects and functions are processes that act on and define relations between sets. In this way we begin to see that we are developing a process-oriented notion of function.

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[5] The number of bijections of a set onto itself is actually n! where n is the cardinality of the set . The cardinality of the set of bijections of a set onto itself is always greater than the cardinality of the set except for the null set and sets of one element. In fact the set of all bijections on the natural numbers is, different from the cardinality of the set of numbers!

Keeping all this in mind, we can increase the generality by considering what it means for two bijections (numbers) to be equal.

Definition of equality.

Consider the set N of all bijective functions from a set onto itself.

If $f_a \in N$ Then f_a is a bijection of the set *a* onto itself

If $f_a, f_b \in N$ Then we say

$f_a \doteq f_b$ if and only if there exists a bijection, e from a *to b*

Informally, we see that if the bijection, e, exists then the sets, a and b, must be the same size or of the same cardinality. And so the functions f_a and f_b are equal if their associated sets are of the same cardinality.

$\doteq$ is an equivalence relation on N

We need to show the following:

1. $f_a \doteq f_a \forall f_a \in N$
2. $f_a \doteq f_b \Leftrightarrow f_b \doteq f_b$
3. If $f_a \doteq f_b$ and $f_b \doteq f_c$ then $f_a \doteq f_c$

The actual proof is trivial and not shown.

Since ≐ is an equivalence relation we can assert that it forms a partition of N into disjoint equivalence classes. Since each equivalence class contains all possible countings of sets "of the same size" we can see that we have arrived at the level of generality in which a given number can count a collection regardless of the nature of the elements of the collection. For this reason we now assert:

Definition of Number

A number is simply any bijection in the equivalence class defined under $\doteq$.

Here the original objection that a number and a set are two very different sorts of things is avoided. In this case we argue that equating number with a kind of function is more natural and intuitive, especially when the particular function is seen as a kindd of counting. In this case the function is seen more as a process, than a completed object. In the following discussion we will define addition.

Addition

We would like to say $f_a + f_b = f_{a \cup b}$. And this would be true if a and b are disjoint. Unfortunately we have no guarantee that this is the case. Russell provides us with a procedure that for any two sets, produces two other sets of the same sizes that are disjoint.[6] This procedure is almost correct. We will describe it and then furnish a correction.

For any two sets, A and B, create another set A* that contains the ordered pair $(\varnothing, a)$ for each element of A and likewise for B create B* with elements of the form $(b, \varnothing)$. Russell asserts that A* and B* are always disjoint. However, if A and B both initially contain the null set, then A* and B* share the element, $(\varnothing, \varnothing)$. It might be thought that we could, modify Russell's construction to avoid this problem in the following way. Every element of A* will be of the form: $(\varnothing, \{a\})$ where $a \in A$ and similarly for B. If $\varnothing \in A, B$, then $(\varnothing, \{\varnothing\}) \in A^* \, and \, (\{\varnothing\}, \varnothing) \in B^*$ and the two sets are disjoint. While this takes care of the case in which $\varnothing$ is in both A and B, the problem returns if $\{\varnothing\}$ is in both A and B. The solution is simple, but

[6] Russell, 68

problematic. Choose some element, c, that is in neither A nor B. Then if $a \in A, B$, then $(c,a) \in A^*$ and $(a,c) \in B^*$. This begs the question, where exactly does c come from? Since is addition is restricted to finite sets, this might not be a problem, but only if the universe of discourse is infinite. It is interesting to note that such a simple idea as addition requires us to consider infinite domains.

From this discussion we can now define addition as:

$$f_a + f_b = f_{a^* \cup b^*}$$

Conclusion

We have developed a possible criticism of Russell's definition of number by arguing whatever a number is; it is not a set. We have provided a possible alternative and argued that it more closely conforms to our intuitive notions about number. This does not argue that Russell's definition is in some sense wrong, but rather that there is another interesting solution to the problem of the definition of number. Both definitions only approach the definition of counting numbers and do not apply to more developed ideas such as the real numbers.